MEASURING THE OPERATIONAL EFFECTIVENESS OF SEARCH AND SCREENING SENSORS FROM EXERCISE DATA

WILLIAM E. CLARK

MEASURING THE OPERATIONAL EFFECTIVENESS

OF SEARCH AND SCREENING SENSORS

FROM EXERCISE DATA

* * * * *

William E. Clark, Jr.

MEASURING THE OPERATIONAL EFFECTIVENESS OF SEARCH AND SCREENING SENSORS FROM EXERCISE DATA

by

William E. Clark, Jr.

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
with major in
MATHEMATICS

United States Naval Postgraduate School
Monterey, California

1 9 6 3

MEASURING THE OPERATIONAL EFFECTIVENESS OF SEARCH AND SCREENING SENSORS FROM EXERCISE DATA

by

William E. Clark, Jr.

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
with major in
MATHEMATICS
from the
United States Naval Postgraduate School

ABSTRACT

At present, methods are well established for measuring the performance of sensors, such as sonars and radars, in an experimental environment. However, the results of these measurements are not necessarily indicative of the sensors' performance under operational conditions. A method is developed, whereby, the operational effectiveness of sensors is determined directly from data obtained from operational exercises. The sensor performance is represented by a lateral range curve called the modified definite range law. Planning errors are discussed and detailed procedures and forms are recommended.

PREFACE

Due to the great differences between experimental and operational environments, measures of the performance of sensors, such as radars and sonars, obtained under experimental conditions are seldom indicative of the sensors' operational performance. For the purposes of operational planning and evaluation of command performance it is desirable to develop a feasible method of measuring the operational effectiveness of sensors and the men who operate and command them. The data base must be the operational exercise.

Herein, such a method is developed. The necessary information is extracted from exercise narratives and navigation charts at post-exercise reconstruction sessions. Later, it is placed on IBM cards for ease of data processing. Periodically, the performance statistics of many sensors can be computed for a variety of operational conditions from this accumulated data. It can then be disseminated to interested commanders.

Included in the development of this method are new definitions of "missed opportunity" and "modified definite range law". Certain probability of detection errors are discussed. In one section detailed procedures and forms are suggested.

The author wished to express his gratitude to Commander Lloyd Bell, United States Navy, for his inspiration and interest in the problem and to Professors W. P. Cunningham and J. R. Borsting, United States Naval Postgraduate School, for their helpful critiques and comments.

TABLE OF CONTENTS

TABLE OF SYMBOLS

$E[Y]$	Expected length of a straight path through a range circle.
$E[d]$	Expected length of path through a range circle.
C	Number of initial detections made by a sensor.
D	The total distance travelled by a sensor within the target's range circle, excluding distance accumulated on successful passes.
M	Number of missed opportunities made by a sensor.
P	Probability of detecting a target.
W	Sweepwidth.
d	Separation of sensors on parallel search legs.
$\bar{P}$	Average probability of detection for two sensors on parallel search legs.
Ep	Percentage probability of detection error between two parallel search plans, using different lateral range curves.
$Var[X]$	Variance of the random variable X.
N	Number of observations of a sensor's performance.

1. Introduction.

Methods are well established for evaluating the performance of sensors such as radars and sonars under controlled experimental environments. The results are useful for comparing the performance of different sensors and for studying the effect of parameters such as sea state on the performance of the sensors. However, due to the controlled experimental nature of the process, the results are usually not in agreement with the performance results of the same sensors under operational conditions.

The problem at hand is to develop a method for evaluating sensor performance under many operational conditions using, as a data base, the information received from the operational exercises which employ the sensors under consideration. It will be clear that it is not only the sensor which is being measured, but also the platform on which it is mounted (aircraft, ship), and the men who operate and command these sensor vehicles. To separately evaluate the performance of the equipment and the performances of the operators and commanders is not an easy task. This procedure does not solve this problem but it does provide for a measure of the effect of crew training and the state of operational readiness on sensor performance.

There are two reasons, at least, for establishing a method for determining the performance of sensors under operational conditions. The first is to provide a cumulative record of the performance of sensors for the purposes

of operational planning and equipment design. The information can be put on IBM cards and stored in a computer facility. As the amount of data becomes large, the actual capabilities of the sensors under various operational conditions and states of crew readiness become well established. Automatic data processing of this data at the computer facility will be routine and the results can be periodically released to interested commands. These results should substantially assist operational planners and equipment designers.

The second reason is to provide an accurate method for evaluating the performance of a sensor during a short exercise such as an operational readiness evaluation where the operational readiness of a commander and his forces is being determined. Not only is this information of basic interest to the commander and his superiors, but it should be as reliable as possible since the results obtained by many commanders is subject to comparison. The procedures to be recommended tend to smooth out the errors inherent in the measurement of results from a small exercise (small amount of data). Also, as the accumulated data, mentioned in the preceding paragraph, becomes large, it is meaningful to compare the results of a small exercise with the previously determined performance figures.

The method recommended here does not require extensive paper work or data collecting by the exercise participants during the conduct of the exercise itself. To do so would

be folly since the extra work may impair the performance of the sensor which is subject to measurement. Only reasonably accurate navigation and usual exercise narratives are required. The performance results are determined from these charts and narratives at the exercise reconstruction session at which all participating units should be represented. Here, information is transferred from the charts and narratives to forms and, later, to IBM cards for delivery to the computer facility.

In Section 2 the expected length of a straight path through the target's range circle of radius R is found to be $\pi R/2$. R is the minimum distance beyond which detection of the target is deemed unlikely. In Section 3 it is shown that this length of path is very close to $\pi R/2$ even if the turns made by a sensor during a search pattern are considered, provided the average length of search legs is greater than the diameter of the range circle. In Section 4 there are two important definitions. The concept of a missed opportunity is discussed and a definition is made. Also, a simple lateral range curve called the modified definite range law is established and said to be representative of the sensor's performance. With these definitions the operational effectiveness of sensors can be determined from available exercise data. In the next three sections the modified definite range law is compared with the definite range law, a normal curve and a triangular curve. Planning prediction errors are determined.

Section 8 outlines the procedures to be used during the exercise reconstruction session and an example of a form in IBM card format on which data is recorded. Section 9 summarizes the procedure and the two appendices outline computer programs used to develop the method.

Who should be interested in using this procedure to evaluate sensor performance? Any command which frequently umpires operational exercises should be if search, screening or reconnaissance is an important factor in the exercise and the sensor-target encounters are similar to those in the following sections. In the Navy, the ASW Defense Forces and Fleet Training Groups are examples that come to mind.

2. Expected Length of a Straight Path Through a Range Circle.

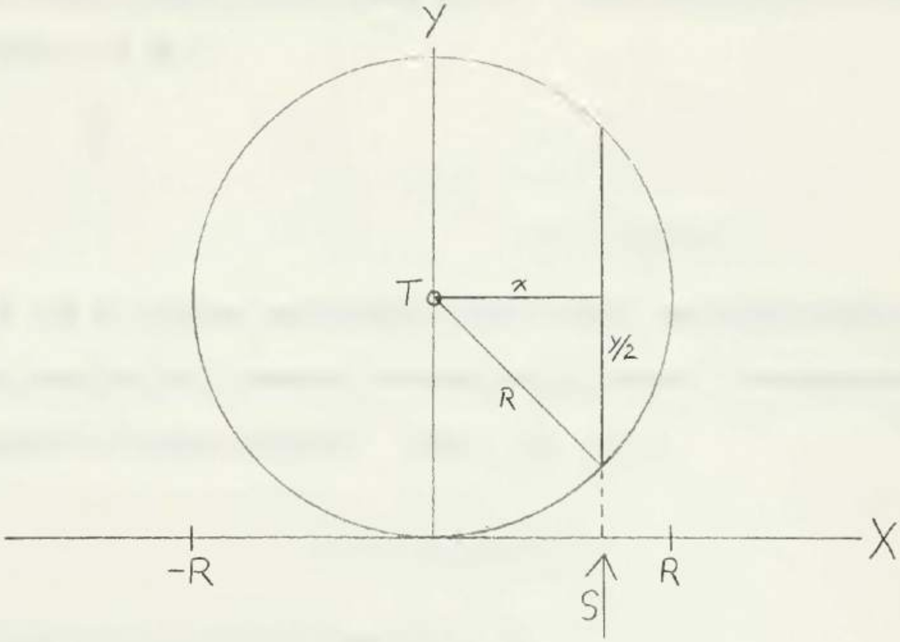


Figure 2.1

Consider the detection situation shown in Fig. 2.1 above. The target, T, is located at the center of a range circle of radius R, the maximum range at which detection is expected to occur. The sensor, S, is in motion and will enter the range circle on a track parallel to the Y axis. The x-coordinate of the sensor's entry point is allowed to vary uniformly from -R to R. Assuming that no turns are made by the sensor, what is the expected length of path through the circle?

Due to the symmetry of the problem, only the right hand semicircle will be considered. Clearly, the expected length of path in either semicircle is equal to that over the entire circle.

Let X be a random variable, the x-coordinate of the sensor's entry point on the range circle, and let X be uniformly distributed from zero to R. The probability density function of X is

$$f_X(x) = 1/R \; ; \qquad 0 < x < R$$
$$\quad = 0 \; ; \qquad elsewhere$$

Let Y be a random variable, the length of path within the range circle of a sensor whose entry point x-coordinate distribution is given above. From Fig. 2.1,

$$y = 2\sqrt{R^2 - x^2}$$

and the Jacobian of transformation is

$$|J|^{-1} = 1 \Big/ \left| dy/dx \right| = \sqrt{R^2 - x^2} \Big/ 2x$$

Therefore, the probability density function of Y is given by

$$g_Y(y) = f_X(x) \, |J|^{-1} = \frac{1}{R} \, \frac{\sqrt{R^2 - x^2}}{2x}$$

but,

$$x = \sqrt{4R^2 - y^2} \Big/ 2$$

therefore,

$$g_Y(y) = \frac{y}{2R\sqrt{4R^2-y^2}} \quad ; \quad 0<y<2R$$

$$= \quad 0 \quad ; \quad elsewhere$$

By the definition of the expected value of a random variable, the expected length of path, $E\left[Y\right]$, is given by

(2.1) $$E[Y] = \int_0^{2R} y' g_Y(y')\,dy' = \frac{1}{2R}\int_0^{2R} y'^2 / \sqrt{4R^2-y'^2}\,dy' = \frac{\pi R}{2}$$

As an example, suppose a sensor is sent on a mission to detect a target, and, due to many factors, the maximum range at which the target can be detected is determined to be ten miles. The target is assumed to be somewhere in an area much larger than that of a ten mile range circle. If the sensor closes to within this maximum detection range, the expected length of path within this range is given by (2.1) with R = 10 miles. That is, $E\left[Y\right] = 15.7$ miles.

In this situation, the assumption of a uniform distribution for the random variable X is valid since there is no reason to prefer one point of entry into the range circle over another. If, however, R is very large or the target's position is more accurately known then the area of search may not be sufficiently large in comparison to the area of the range circle to allow this assumption.

In other detection situations only the target or both the sensor and the target may be in motion. This is of no

consequence to the solutions presented since only relative motion need be considered. In most practical situations either the sensor or the target moves very slowly relative to the other and only a small error is incurred by fixing the slower one.

Another assumption made in this section is not at all realistic. Here, the sensor is not allowed to turn during the passage through the range circle. In the next section the sensor will be allowed to make turns and the expected length of path within the range circle will be investigated. The results will be compared with those of this section.

3. Expected Length of Path Through a Range Circle, Allowing Turns.

Let $E\left[d\right]$ be the expected length of path through a range circle. In the previous section it was shown that $E\left[d\right] = E\left[Y\right] = \pi R/2$ when no turns by the sensor are allowed. In order to determine $E\left[d\right]$ when the sensor is allowed to turn, it is necessary to know the search pattern of the sensor. For a particular mission this may be know, but due to navigation errors or deliberate changes of plans, it is often not possible to continue it. In any case, search plans vary with time, location, weather, and operational situations making it infeasible to choose one or even several of them to determine $E\left[d\right]$. The method used here will be to the simulation of search patterns by probability distributions with variable parameters and subsequent approximation of $E\left[d\right]$ by computerized wargame techniques.

Search patterns which are composed of a series of straight tracks can be characterized by two random variables, the length of search legs, V, and the amount of turn, A, between successive legs. The density function of each is assumed to be triangular due to ease of computation and proximity to the normal density function.

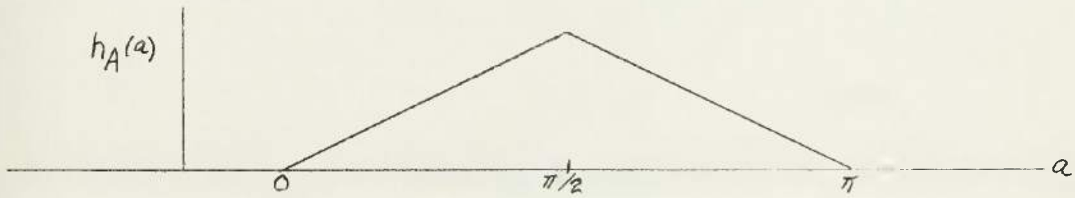


Figure 3.1

Assuming that the most likely turn is one of 90 degrees, the probability density function for the amount of turn, A, is given by

(3.1)

$$\begin{aligned} h_A(a) &= 4a/\pi^2 \quad ; \quad 0 < a < \pi/2 \\ &= 4(\pi - a)/\pi^2 \quad ; \quad \pi/2 \le a \le \pi \\ &= 0 \quad ; \quad \text{elsewhere} \end{aligned}$$

This density function is shown in Fig. 3.1. Note that the function is defined on π radians, therefore it is necessary for the program to determine if a turn is to be made to the right or to the left.

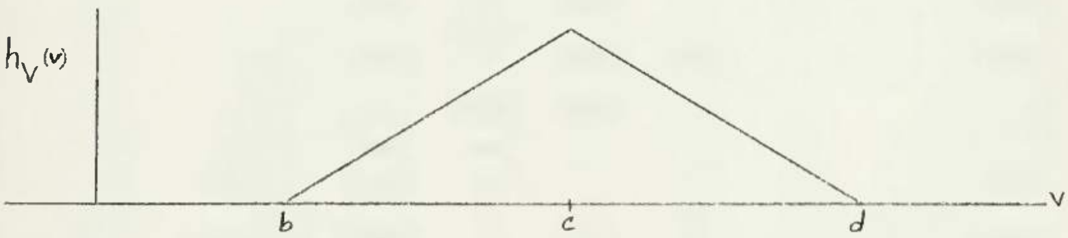


Figure 3.2

Similarly, the probability density function of V, the length of search legs, is shown in Fig. 3.2 where v is allowed to vary from v = b to v = d with mean value at v = c. Also, (c-b) = (d-c). The density function is

(3.2)

$$\begin{aligned} h_V(v) &= (v-b)/(c-b)^2 \quad ; \quad b \le v \le c \\ &= (d-v)/(c-b)^2 \quad ; \quad c < v \le d \\ &= 0 \quad ; \quad \text{elsewhere} \end{aligned}$$

The function is completely specified by (d-b) and the mean value, c. These are allowed to vary by the program.

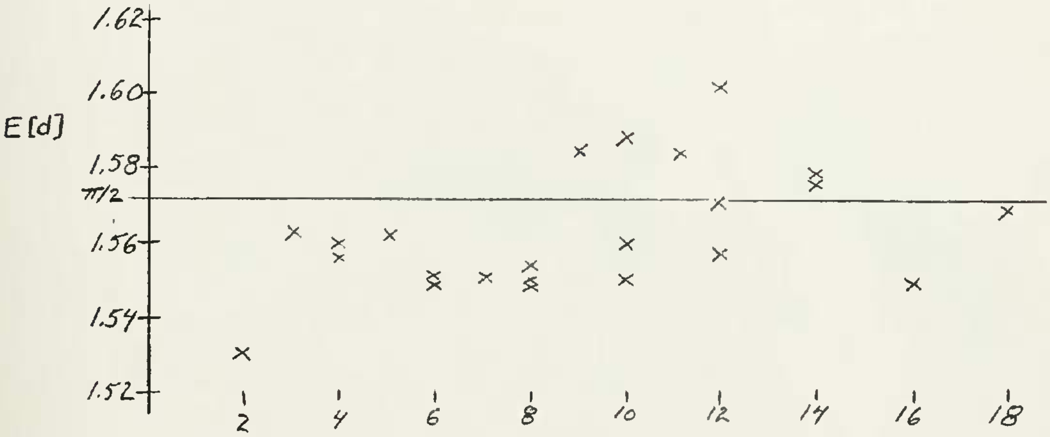
For each pair of values chosen for c and (d-b), 1000 runs on the target were made by the sensor to determine $E\left[d\right]$. The program itself is outlined in Appendix A and the results are tabulated in Table 3.1 below. Note that throughout the program, R is one for simplicity.

Table 3.1

c	(d-b)	Number of runs having below indicated number of turns						$E\left[d\right]$
		0	1	2	3-5	6-10	11-100	
1/64	1/32	0	331	164	160	106	239	.239
1/32	1/16	0	351	149	180	84	236	.461
1/16	1/8	0	312	153	164	92	279	.953
1/8	1/4	1	363	182	161	77	216	1.189
1/4	1/2	2	375	162	167	114	180	1.520
1/2	1	14	428	187	218	128	25	1.441
1	2	59	553	236	142	10	0	1.500
2	4	237	656	94	13	0	0	1.529
3	2	490	510	0	0	0	0	1.562
4	4	591	409	0	0	0	0	1.595
4	8	547	440	12	1	0	0	1.582
5	2	688	312	0	0	0	0	1.562
6	4	743	257	0	0	0	0	1.549
6	8	718	282	0	0	0	0	1.548
7	2	792	208	0	0	0	0	1.551
8	4	793	207	0	0	0	0	1.552
8	8	776	224	0	0	0	0	1.575
8	16	744	255	1	0	0	0	1.551
9	2	819	181	0	0	0	0	1.585
10	4	832	168	0	0	0	0	1.549
10	8	855	145	0	0	0	0	1.559
10	16	827	173	0	0	0	0	1.589
11	2	849	151	0	0	0	0	1.576
12	4	877	123	0	0	0	0	1.604
12	8	873	127	0	0	0	0	1.554

12	16	858	142	0	0	0	0	1.569
14	8	872	128	0	0	0	0	1.576
14	16	872	128	0	0	0	0	1.576
16	16	890	110	0	0	0	0	1.542
18	16	928	72	0	0	0	0	1.562

From the program results, summarized in Table 3.1, it can be seen that for $c \geq 2$ the approximated value for $E\left[d\right]$ is within 2.7% of $E\left[Y\right] = 1.571$. For $c \leq 1/8$, $E\left[d\right]$ is much smaller than $E\left[Y\right]$ and decreases rapidly as c decreases. The values of $E\left[d\right]$ obtained by the program are plotted below for values of $c \geq 2$.



Mean length of search leg, c

Figure 3.3

The average value of $E\left[d\right]$ for $c \geq 2$ is 1.565 which is sufficiently close to 1.571 to conclude that when the mean value of the length of search leg is greater than the diameter of the range circle, $E\left[d\right] = E\left[Y\right] = \pi R/2$.

Since there are many actual detection situations where $c \geq 2R$ and since $E\left[d\right]$ is a function of c for $c < 2R$, only the case of $c \geq 2R$ will be considered here. Now that the effect of various search plans on the value $E\left[d\right]$ is known, a method for measuring the operational effectiveness of sensors will be developed.

4. Measuring the Operational Effectiveness of Sensors.

The definitions and procedures of this section are applicable to operational detection situations where the following conditions hold:

(a) There exists a maximum range, R, beyond which detection of the target by the sensor under consideration is deemed unlikely.

(b) The mean length of sensor search legs is greater than 2R, the diameter of the target's range circle.

(c) The operation or exercise must provide the data necessary for the determination of the sensor's effectiveness.

The first two requirements have been discussed in the previous sections. The major sources of data from exercises are the reconstructed navigation charts of the sensors and targets. It is also important to record other factors which are likely to affect the ability of the sensor to detect such as sea state and visibility. If the sensor's track and that of the target are drawn to the same scale, by superimposing one over the other, it becomes an easy matter to determine the total distance travelled by the sensor within the target's range circle. Recommended procedures will be presented in detail in Section 9, but presently, it is enough to realize that it is feasible to gather meaningful data from the track charts of sensors and targets.

Suppose that considerable data has been collected from one or more exercises and we wish to evaluate the performance of a particular type of sensor in a given sensor-target situation. The following information is known:

(a) C - the number of initial detections made by the sensor.

(b) D - the total distance travelled by the sensor within the target's range circle, excluding distance accumulated on successful passes.

The following definition of a missed opportunity for detection is crucial in this development of a measure of effectiveness.

A <u>missed opportunity</u> is defined as equivalent to a distance $E\left[d\right] = \pi R/2$ travelled by the sensor within the target's range circle during one or more unsuccessful passes through the range circle. The total number of missed opportunities, M, accumulated by the sensor during the evaluation period is therefore

(4.1) $M = 2D/\pi R$

The sensor's probability, P, of detecting the target given that he enters the range circle is given by

(4.2) $P = C/M+C$

which is simply the number of successes divided by the number of opportunities. This probability is defined over a path width of 2R giving the following lateral range curve.

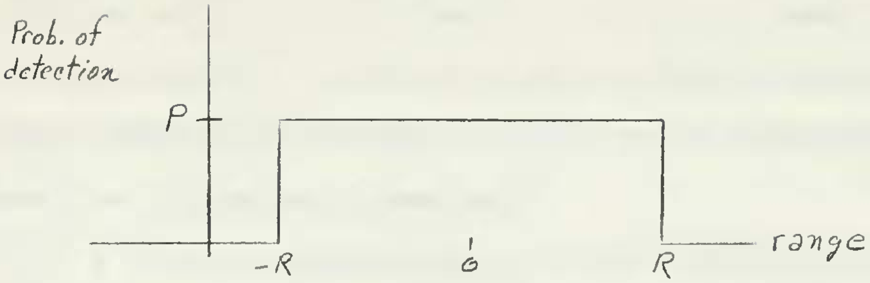


Figure 4.1

This is a modified definite range law, differing from the definite range law in that P need not equal one.

As an example of the above, suppose that during an exercise a particular radar was being used to detect small fishing boats in a sea state of 3. Assume R = 40 miles, C = 10, and D = 157 miles. From (4.1) the number of missed opportunities, M, is determined to be

$$M = 157/(1.57 \times 40) = 2.5$$

The probability of detection, P, is

$$P = 10/(2.5 + 10) = 0.8$$

and the lateral range curve is given below.

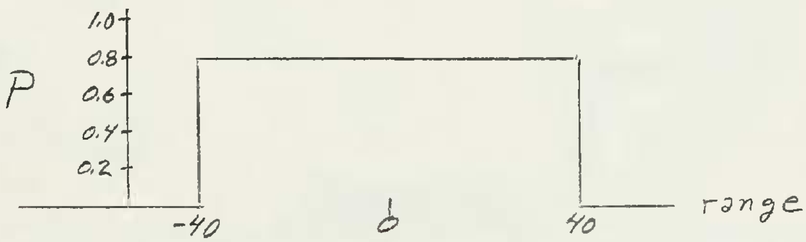


Figure 4.2

In the subsequent sections, the above type of lateral range curve will be compared with other curves to determine possible sources of error and to demonstrate the utility of the modified definite range law with respect to the operational evaluation of sensors.

5. Modified Definite Range Law vs. Definite Range Law.

Another method of measuring the effectiveness of sensors makes use of the definite range law. This method involves the following parameters:

(a) R - the maximum range at which detection is expected to occur.

(b) C - the number of initial contacts by the sensor.

(c) Op - the total number of opportunities.

The terms, R, and C, are identical to those used in previous sections to develop the modified definite range law. Often, in present practice, the number of opportunities, Op, is obtained by counting the number of times the sensor penetrated the range circle, with no regard to the opportunity distance accumulated within this range. In other words, each of the following two tracks would result in one missed opportunity by this definition.

Figure 5.1

Clearly S_2 had a much greater opportunity to detect the target, T, then did S_1, and this difference should be reflected in the sensor performance evaluation. If the number of opportunities is very large, the effect of extreme examples such as those shown in Fig. 5.1 diminishes and

the number of opportunities obtained by this counting method approaches the value (M + C) of equation (4.2). However, if the number of opportunities is small, as it would be in a small exercise, the use of the definition of missed opportunity given in Section 4 will result in a truer evaluation of the sensor's performance. This is especially important if sensor performance figures obtained from small exercises are used in part to evaluate and compare the operational readiness of military commanders. Several unsuccessful passes of the type of Fig. 5.1 (a) would be recorded as only one or two missed opportunities according to Section 4, while a pass such as that in Fig. 5.1(b) would be equivalent to about three missed opportunities.

The next step in the definite range method is to define the sweepwidth, W, as the width of a path centered at the sensor in which the probability of detection is one. The value of W is given by

$$(5.1) \quad W = 2RC/Op.$$

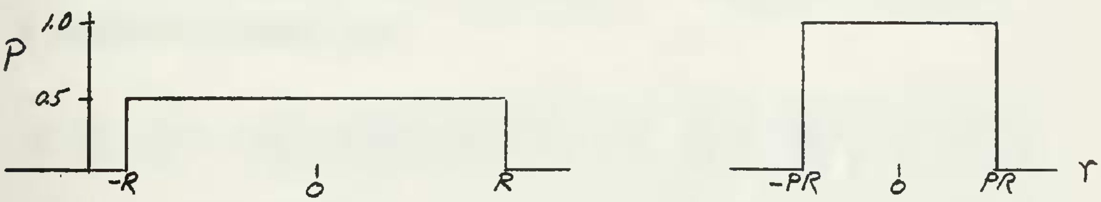
The sweep width, W, is often quoted as a measure of the performance of a sensor.

Although the modified definite range law and the definite range law are similar, there are two reasons for preferring the former. The modified law is a more accurate representation of the sensor's actual performance. The probability of detection, P, is determined empirically over a path whose width is equal to 2R. If P is determined

from a large amount of data it can be predicted quite confidently that the sensor will continue to detect targets with probability P under the same operating conditions. In the case of the definite range law, however, the probability of detection is arbitrarily set at one and the resulting sweep width does not necessarily represent the actual range of the sensor. The disparity between the two lateral range curves increases as P approaches zero, and in the extreme case of P = 1, they are identical. Given that P is the probability of detection over a path of width 2R in the modified law, the sweep width, W, for the equivalent definite range law is

$$W = 2PR$$

The two lateral range curves are shown below for the case where P = .5.

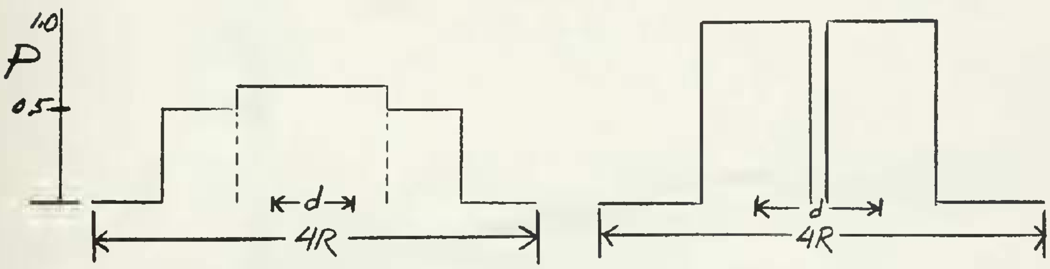


Modified definite range law   Definite range law

Figure 5.2

The second reason for preferring the modified law is applicable when two or more sensors move along parallel tracks. Consider a strip of width 4R within which, the lateral range curves of two parallel tracking sensors are contained. Let $\overline{P}$ be the average probability of detection measured over the width of the strip. It can be seen that

$\overline{P}$ is a function of d, the separation of the sensors. Fig. 5.3 below depicts each range law. Here d is such that overlap of the lateral range curves occurs only in the case of the modified definite range law. When d is less than 2PR the lateral range curves for the definite range law will overlap also.



Modified definite range law Definite range law

Figure 5.3

Let $\overline{P}_m$ be the average probability of detection for two sensors obeying the modified definite range law in a strip of width 4R.

(5.2) $$\overline{P_m} = \frac{2dP + (2R-d)(2P-P^2)}{4R} = P - \frac{P^2}{2} + \frac{dP^2}{4R}; \quad 0 \le d \le 2R$$

Let $\overline{P}_d$ be the average probability of detection resulting from the equivalent definite range law, averaged over the same width, 4R.

(5.3) $$\overline{P_d} = \frac{4PR}{4R} = P \quad ; \quad 2PR \le d \le 2R$$

$$= \frac{P}{2} + \frac{d}{4R} \quad ; \quad 0 \le d \le 2PR$$

In deriving the above equations, the following formula was used to determine the probability of detection in the area of overlap, Pc, assuming independence.

$$Pc = 1-(1-P)^2 = 2P-P^2$$

Below is a plot of $\bar{P}m$ and $\bar{P}d$ for P = .6. Note that the maximum differences occur at d = 0 and at d = 2PR = 1.2R.

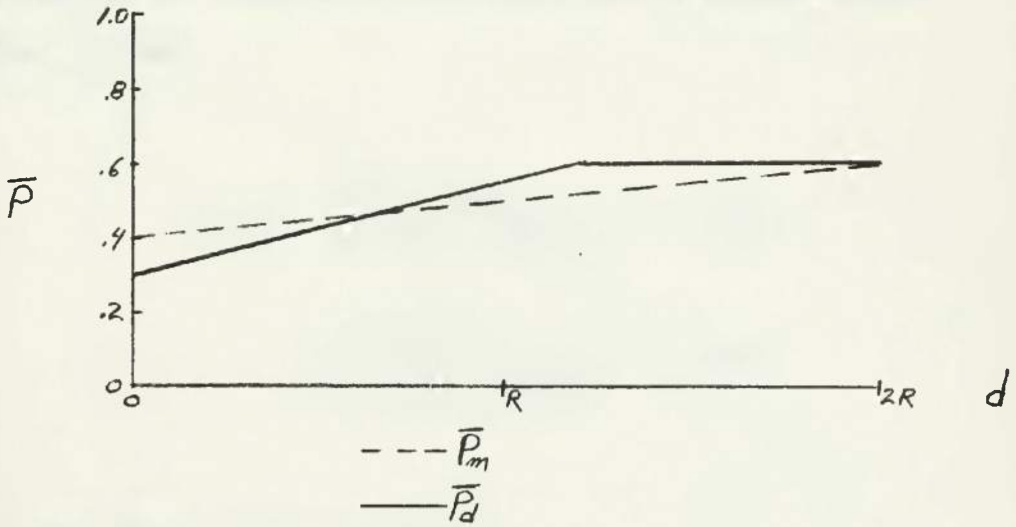


Figure 5.4

For 0<P<1 the plot of $\bar{P}m$ and $\bar{P}d$ as a function of d is similar to Fig. 5.4 with the change in slope of the $\bar{P}d$ curve occurring at d = 2PR. In the extreme cases, where P = 0 and P = 1, $\bar{P}m = \bar{P}d$ for $0 \leq d \leq 2R$.

The average probability difference $E = \bar{P}m - \bar{P}d$ and the percentage error, Ep, is defined as

$$E_P = \frac{2|E|}{\bar{P}_m + \bar{P}_d} \times 100$$

From (5.2) and (5.3),

$$E = \frac{P^2}{2}\left(\frac{d}{2R} - 1\right) \qquad ; \quad 2PR \le d \le 2R$$

$$= \frac{P}{2}(1-P) + \frac{d}{4R}(P^2-1) ; \qquad 0 \le d \le 2PR$$

Note that at d = 2RP/(1+P) the value of E changes sign.

For $2PR \le d \le 2R$;

(5.4)
$$E_P = \frac{P^2(1 - d/2R)}{2P - P^2/2 + dP^2/4R} \times 100$$

$$= \frac{P^2(4R-2d)}{P^2(d-2R) + 8PR} \times 100$$

(5.5) For $2PR/(1+P) \le d \le 2PR$

$$E_P = \frac{P^2(4R-2d) + 2d - 4RP}{P^2(d-2R) + 6RP + d} \times 100$$

and for $0 \le d \le 2PR/(1+P)$

(5.6)
$$E_P = \frac{P^2(2d-4R) + 4RP - 2d}{P^2(d-2R) + 6RP - d} \times 100$$

As can be seen from Fig. 5.4, the maximum errors occur at d = 0 and at d = 2PR. Setting d = 0 in equation (5.6)

(5.7) $$E_P = \frac{2(1-P)}{(3-P)} \times 100 \, ; \quad d = 0$$

A plot of (5.7) is shown below.

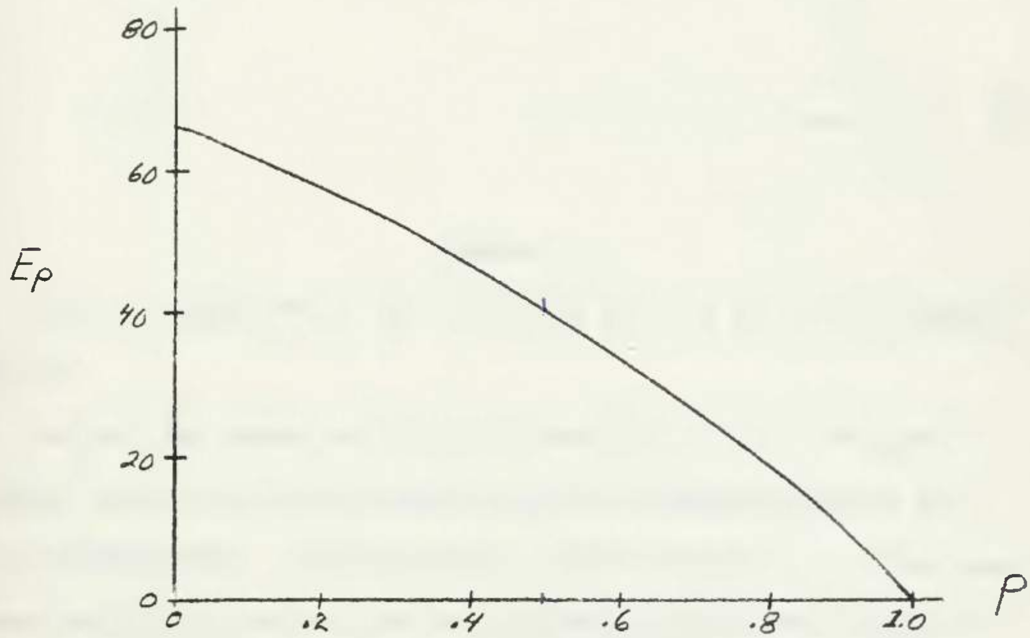


Figure 5.5

At the other point of maximum error, d = 2PR, Ep is obtained by setting d = 2PR in equation (5.4).

(5.8) $$E_P = \frac{2P(1-P)}{P^2-P+4} \quad ; \quad d = 2PR$$

A plot of (5.8) is shown below.

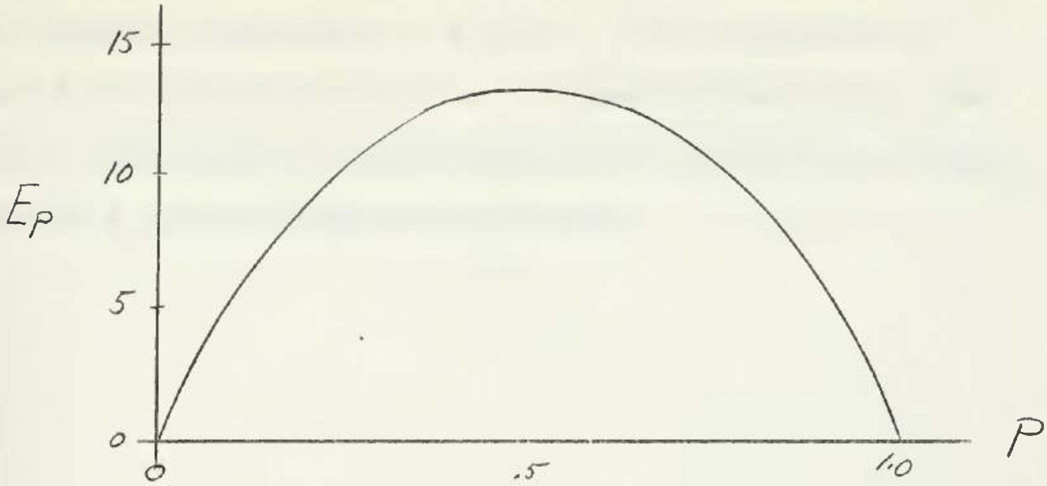


Figure 5.6

At $d = 2PR/(1+P)$, $\bar{P}m = \bar{P}d$ and $Ep = 0$ for all values $0 \leq p \leq 1$.

As can be seen in Fig. 5.5 and Fig. 5.6, the percentage error is most significant for cases where $P \leq .5$ and $d < 2PR/(1+P)$. This implies that for small P, the use of the definite range law for planning purposes will result in a large error in the prediction of $\bar{P}$ if the track spacing d is small and if the modified lateral range curve accurately represents the performance of the sensor.

If the sensors are spaced such that the lateral range curves of the definite range law are adjacent, $d = 2PR$, and Fig. 5.6 gives the percentage error as a function of P, assuming again that the sensor obeys the modified law. The maximum error of 13.3% occurs at $P = .5$.

The sweepwidth concept is useful, however, since it allows for the measurement of the capability of a sensor by a single value, W, while the modified definite range

law requires knowledge of P and R. The sweepwidth, W, should be considered only as a figure of merit for the sensor and should not necessarily be considered as representing a valid lateral range curve.

6. Modified Definite Range Law vs. Normal Curve.

It may be argued that both the definite range law and the modified definite range law are unrealistic since it is highly unlikely that there is a range, R, at which the probability of detection drops sharply from P to zero. A lateral range curve may be derived from physical assumptions concerning the operational environment whereby the probability is a function of the lateral range, r, rather than a constant. Quite likely, P will approach zero monotonically as $|r|$ approaches infinity. Here, the normal lateral range function will be defined and chosen as representative of such a family of functions.

Suppose that a lateral range curve for a sensor has been obtained in accordance with the modified definite range law method of Section 4. If, however, the true lateral range curve has the shape of the normal probability density function, what errors are incurred by assuming the modified definite range law holds?

Since a lateral range curve is not a true probability density function, it will be necessary to apply a transformation to it in order to apply the theory of probability to its use. The following conventions will be adopted throughout this section. The random variable subscript used in the function notation of Section 2 will be dropped, i.e.,

$$f_X(x) = f(x)$$

Also, if two functions have the same shape, but one is a probability density function and the other is a lateral range function, the prime superscript will be used to identify the latter. Two functions, f and f', have the same shape if

$$\frac{f'(r)}{f(x)} = \frac{r}{x} = k \; ; \quad \text{where } k \text{ is constant.} \tag{6.1}$$

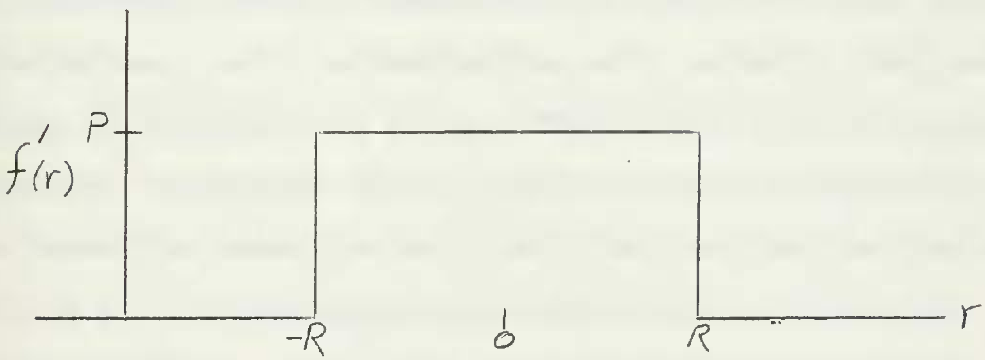


Figure 6.1

Let f'(r) be the function represented by the lateral range curve of the modified definite range law, Fig. 6.1.

$$\begin{aligned} f'(r) &= P \; ; \quad -R < r < R \\ &= 0 \; ; \quad \text{elsewhere} \end{aligned} \tag{6.2}$$

This function is not a probability density function unless the area 2RP = 1. To transform f'(r) to the uniform probability density function f(x) having the same shape, let $k = \sqrt{2RP}$ in equation (6.1).

$$f(x) = \sqrt{P/2R} \quad ; \quad -\sqrt{R/2P} < x < \sqrt{R/2P}$$

$$= 0 \quad ; \quad \text{elsewhere}$$

One of the assumptions made in the development of the modified definite range law was that the probability of detecting the target from outside the range circle was very small, and is indeed set to zero as in Fig. 6.1. If the normal curve is assumed to hold, however, the probability of detection is greater than zero for all ranges, therefore, a decision must be made as to how much of the area under the normal curve should fall within the limits $-R < r < R$ of the modified law in order not to invalidate this assumption. Two normal curves will be considered with 95% and 99% of the area being within these limits.

Let $f_1'(r)$ be a normal lateral range function such that

$$\int_{-R}^{R} f_1'(r)\,dr = .95 \times 2RP$$

and let $f_2'(r)$ be a similar function such that

$$\int_{-R}^{R} f_2'(r)\,dr = .99 \times 2RP$$

$f_1(x)$ and $f_2(x)$ are normal probability density functions

having the same shape as $f_1'(r)$ and $f_2'(r)$ respectively and, furthermore,

$$f_1'(r) = \sqrt{2RP}\; f_1(x)$$

$$f_2'(r) = \sqrt{2RP}\; f_2(x)$$

Since $f_1(x)$ and $f_2(x)$ have the same relation to the uniform function, $f(x)$, as $f_1'(r)$ and $f_2'(r)$ have to $f'(r)$, the functional form of $f_1(x)$ and $f_2(x)$ can be determined. Let the mean of all functions be zero.

$f_1(x)$ is normal with mean 0 and variance σ_1^2 such that

$$\int_{-\sqrt{R/2P}}^{\sqrt{R/2P}} f_1(x)\,dx = .95$$

therefore $$\sigma_1 = \frac{1}{1.96}\sqrt{R/2P}$$

and $$f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\, e^{-\frac{x^2}{2\sigma_1^2}} \; ; \quad -\infty < x < \infty$$

Similarly, $$f_2(x) = \frac{1}{\sqrt{2\pi}\,\sigma_2}\, e^{-\frac{x^2}{2\sigma_2^2}} \; ; \quad -\infty < x < \infty$$

where $$\sigma_2 = \frac{1}{2.57}\sqrt{R/2P}$$

Finally, by (6.1),

$$f_1'(r) = \sqrt{2RP}\ f_1(x) \quad ; \quad -\infty < r < \infty$$

$$f_2'(r) = \sqrt{2RP}\ f_2(x) \quad ; \quad -\infty < r < \infty$$

Fig. 6.2 is a graph of these two normal lateral range functions and the modified definite range law.

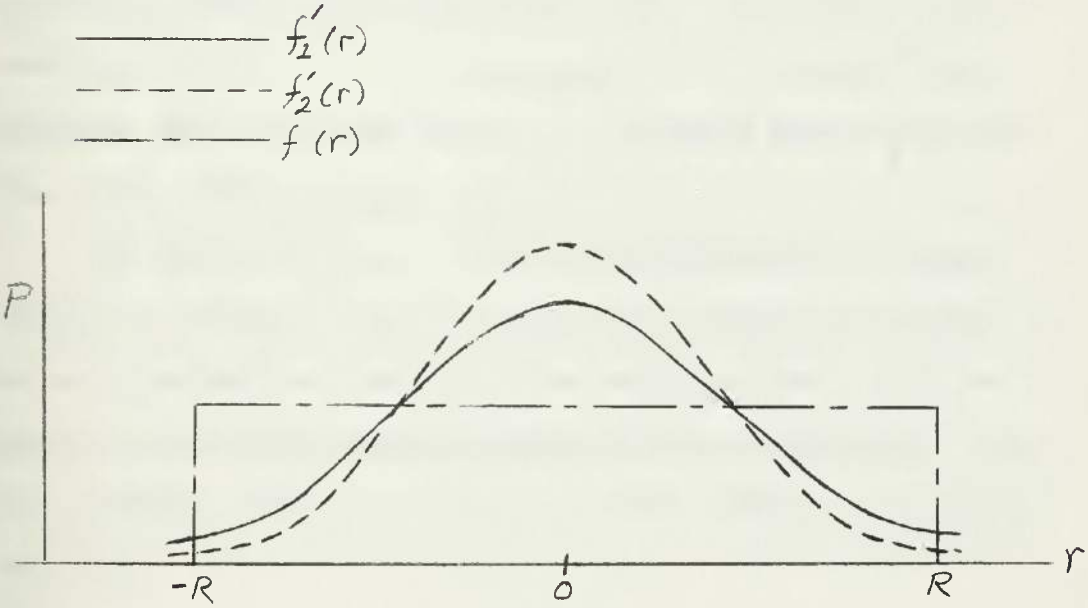


Figure 6.2

Since the maximum allowable value for the probability of detection is one, there are also maximum allowable values of P in (6.2) in order that the maximum values of $f_1'(r)$ and $f_2'(r)$ do not exceed one. The maximum values of $f_1'(r)$, $f_2'(r)$, $f_1(x)$ and $f_2(x)$ occur at r = 0 and x = 0. Therefore for $f_1'(r)$

$$f_1'(0) = \sqrt{2RP}\ f_1(0) = \frac{(1.96)(2P)}{\sqrt{2\pi}} = 1$$

$$P = \frac{\sqrt{2\pi}}{3.92} = .639$$

and the normal lateral range function $f_1'(r)$ can apply only when the probability of detection in the modified definite range law is less than .639. Similarly, by setting $f_2'(0) = 1$, it is found that P is limited to a maximum value of .488 when it is assumed that $f_2'(r)$ is the true lateral range curve.

In general, let α be the percentage of the area under the normal lateral range curve which is allowed to be outside the limits of the modified law, and let Pmax be the associated maximum allowable value of P in the modified definite range function. Pmax is related to α by

$$P_{max} = \frac{1}{t}\sqrt{\frac{\pi}{2}} \qquad \text{where}$$

$$\frac{1}{\sqrt{2\pi}} \int_0^t e^{-\frac{u^2}{2}}\, du = \frac{1-\alpha}{2}$$

The value, t, can be obtained from a table of the normal probability function. A plot of Pmax as a function of α is shown below.

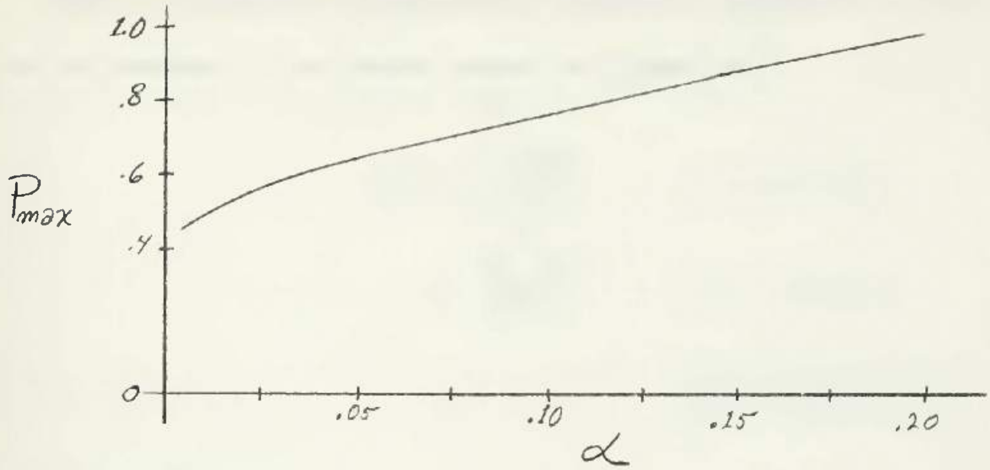


Figure 6.3

Since the normal function is positive for all finite values of its argument, it will not be compared with the modified definite range law directly to determine the errors incurred when the sensors are searching on parallel tracks. In the next section, the normal function will be approximated by the triangular distribution and the parallel tracking error will be determined.

7. Triangular Lateral Range Function and Prediction Errors.

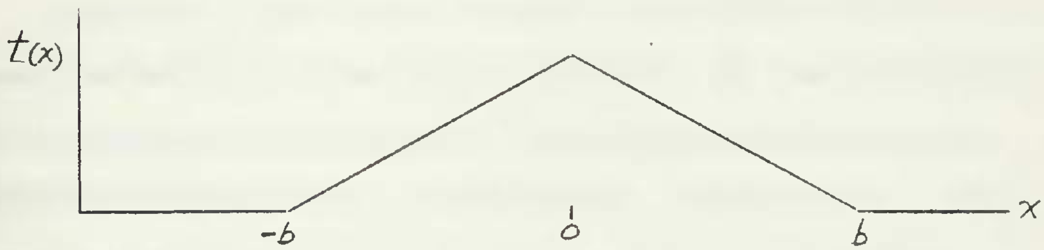
The triangular probability density function, t(x), with parameter b and mean zero is given by

$$t(x) = \frac{x+b}{b^2} \;; \quad -b \le x \le 0$$

$$= \frac{b-x}{b^2} \;, \quad 0 < x \le b$$

$$= 0 \;; \quad \text{elsewhere}$$

and is shown below.



Triangular density function

Figure 7.1

Since $E[x] = 0$, $Var[x] = E[x^2] = \int_{-b}^{b} x^2 \, t(x) \, dx$

$$Var[x] = \frac{1}{b^2}\int_{-b}^{0}(x^3 + bx^2)\,dx + \frac{1}{b^2}\int_{0}^{b}(bx^2 - x^3)\,dx$$

$$= \frac{b^2}{6}$$

As can be seen in Fig. 7.2 below, the triangular distribution is a good approximation of the normal function. Both functions have $E[X] = 0$ and $Var[X] = 1$.

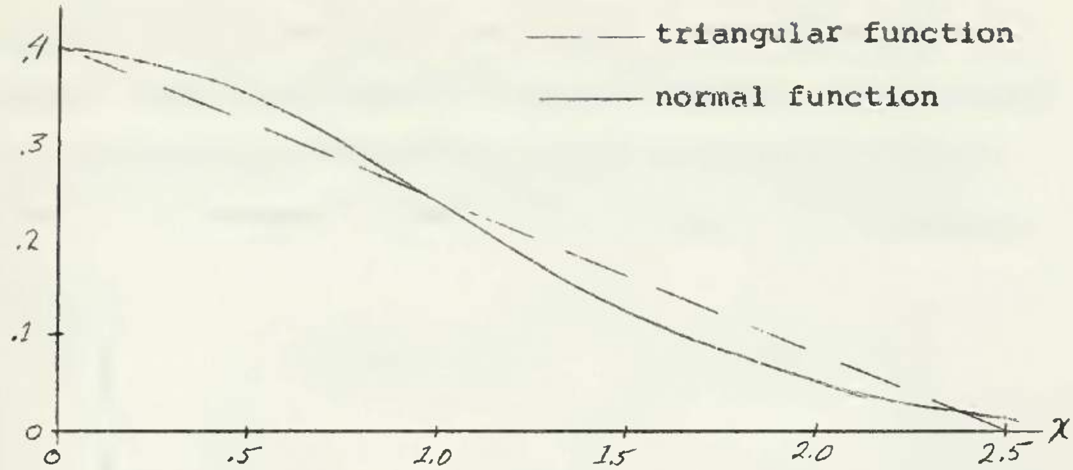


Figure 7.2

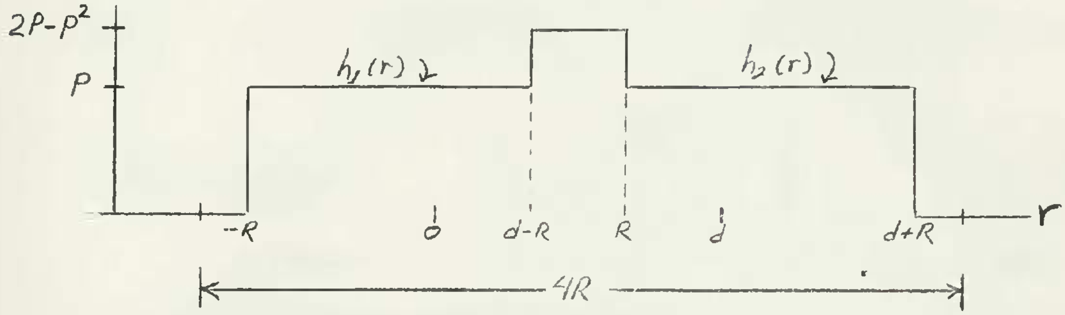
Similarly, the normal lateral range function may be approximated by a triangular function. If the probability of detection in the modified definite range law is not greater than one half, the triangular function can be assumed to hold satisfying the following two conditions.

(a) The probability of detection is equal to zero for $R < r < -R$.

(b) The maximum probability of detection is not greater than one.

The percent prediction error, Ep, incurred in parallel track searches will now be determined in a manner similar to that in Section 5. Here, however, the modified definite range law will be compared with the triangular lateral range function, t(r), realizing that the latter is a good approximation of the normal lateral range function. Consider two sensors, searching on parallel tracks of spacing d, and whose lateral range curves are confined to a strip

of width 4R. Let $t_1(r)$ and $t_2(r)$ be triangular lateral range functions and $h_1(r)$ and $h_2(r)$ be modified definite lateral range functions for the two sensors. The problem is to determine the prediction error incurred when one function is assumed to hold when in fact the other does.



Overlapping modified definite lateral range functions

Figure 7.3

For the case of the modified law, Fig. 7.3, $P \leq .5$ and

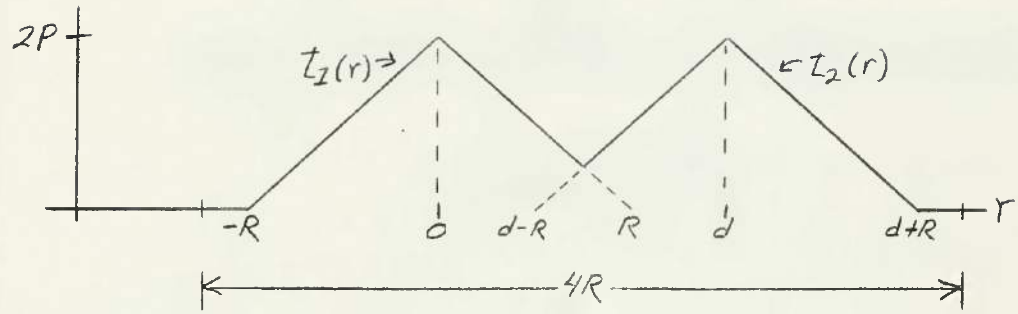
$$
\begin{aligned}
h_1(r) &= P \quad ; \quad -R < r < R \\
&= 0 \quad ; \quad \text{elsewhere} \\
h_2(r) &= P \quad ; \quad d-R < r < d+R \\
&= 0 \quad ; \quad \text{elsewhere}
\end{aligned}
$$

The average probability of detection, $\bar{P}m$, over the strip of width 4R is given by

(7.1) $$\bar{P}_m = \frac{1}{4R}\left[2\int_{-R}^{d-R} h_1(r) + \int_{d-R}^{R} \left(h_1(r) + h_2(r) - h_1(r)h_2(r)\right)\right]$$

$$= \frac{1}{4R}\left[2dP + (2R-d)(2P-P^2)\right]$$

$$= \frac{1}{4R}\left(4RP - 2RP^2 + dP^2\right)$$



Overlapping triangular lateral range functions.

Figure 7.4

For the case of overlapping triangular lateral range functions, Fig. 7.4,

$$t_1(r) = 2P\left(\frac{r+R}{R}\right); \qquad -R < r < 0$$

$$= 2P\left(\frac{R-r}{R}\right); \qquad 0 < r < R$$

$$= 0 \qquad ; \qquad \text{elsewhere}$$

$$t_2(r) = 2P\left(\frac{r+R-d}{R}\right); \qquad d-R < r < d$$

$$= 2P\left(\frac{d+R-r}{R}\right); \qquad d < r < d+R$$

$$= 0 \qquad ; \qquad \text{elsewhere}$$

The average probability of detection, $\bar{P}t$, over the strip of width 4R is given by

(7.2)

$$\bar{P}_t = \frac{1}{4R}\left[2\int_{-R}^{0} t_1(r) + 2\int_{0}^{d-R} t_1(r) + \int_{d-R}^{R}\left\{t_1(r) + t_2(r) - t_1(r)\,t_2(r)\right\}\right]$$

$$= \frac{1}{4R}\left[2\int_{-R}^{0} 2P\left(\frac{r+R}{R}\right) + 2\int_{0}^{d-R} 2P\left(\frac{R-r}{R}\right) + \int_{d-R}^{R}\left\{2P\left(\frac{R-r}{R}\right) + 2P\left(\frac{r+R-d}{R}\right) - 4P^2\left(\frac{R-r}{R}\right)\left(\frac{r+R-d}{R}\right)\right\}\right]$$

for $R \le d \le 2R$. For $0 \le d \le R$, $\bar{P}t$ is given by

(7.3)

$$\bar{P}_t = \frac{1}{4R}\left[2\int_{-R}^{d-R} t_1(r) + 2\int_{d-R}^{0}\left\{t_1(r) + t_2(r) - t_1(r)\,t_2(r)\right\} + \int_{0}^{d}\left\{t_1(r) + t_2(r) - t_1(r)\,t_2(r)\right\}\right]$$

$$= \frac{1}{4R}\left[2\int_{-R}^{d-R} 2P\left(\frac{r+R}{R}\right) + 2\int_{d-R}^{0}\left\{2P\left(\frac{r+R}{R}\right) + 2P\left(\frac{r+R-d}{R}\right) - 4P^2\left(\frac{r+R}{R}\right)\left(\frac{r+R-d}{R}\right)\right\}\right.$$
$$\left. + \int_{0}^{d}\left\{2P\left(\frac{R-r}{R}\right) + 2P\left(\frac{r+R-d}{R}\right) - 4P^2\left(\frac{R-r}{R}\right)\left(\frac{r+R-d}{R}\right)\right\}\right]$$

Let $E = \bar{P}t - \bar{P}m$ and define the percentage error as in Section 5.

(7.4)

$$E_p = \frac{2\,|E|}{\bar{P}_t + \bar{P}_m} \times 100$$

A small program was written to determine Ep for various values of P and d. See Appendix B for the program and complete results.

Ep is plotted below as a function of d for P = .5 and P = .3.

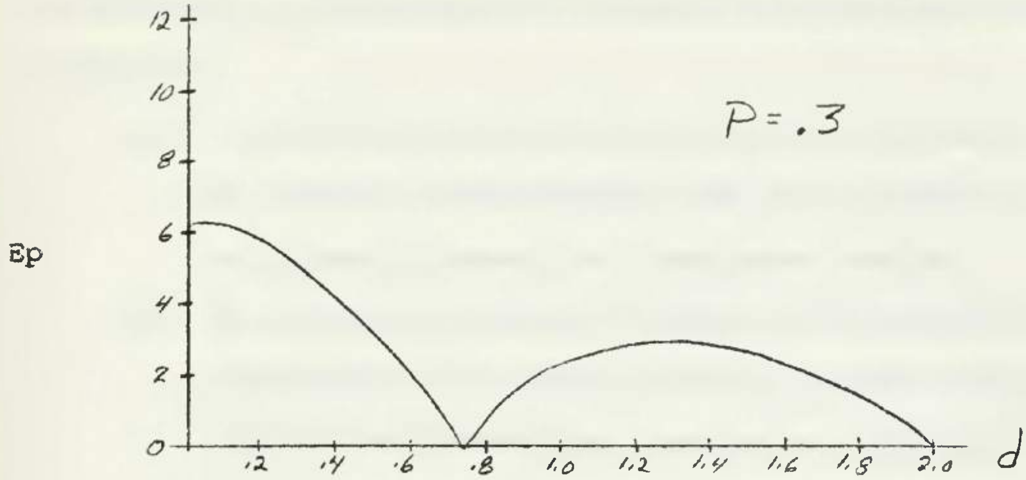


Figure 7.5

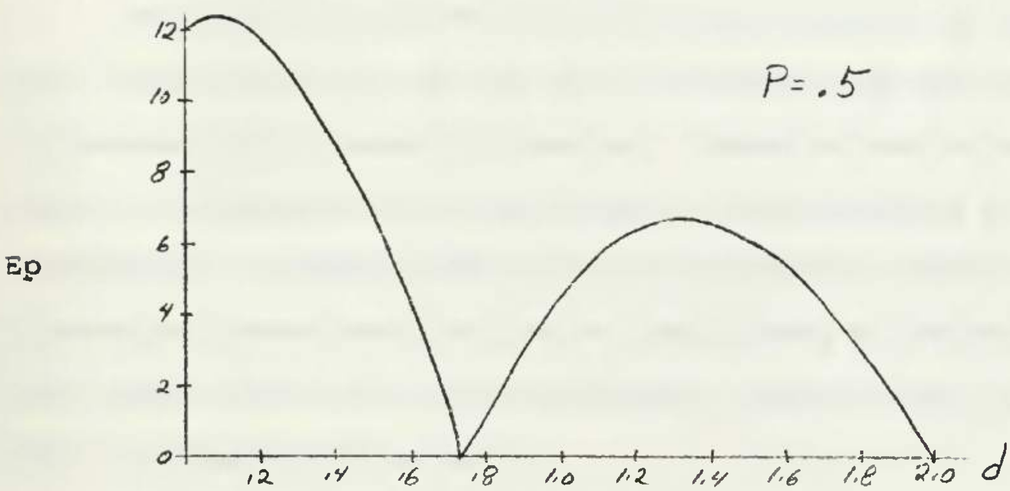


Figure 7.6

8. Suggested Procedures and Forms.

There are two reasons for establishing a method for determining the performance of sensors under operational conditions.

(a) To provide a cumulative record of the performance of sensors (and targets) for the purposes of operational planning and equipment design.

(b) To provide an accurate method for evaluating the performance of a sensor during a short exercise such as an operational readiness evaluation where the operational readiness of a commander and his forces is being determined.

The degree of success of the method depends on the type and reliability of the data collected and the data processing and storage procedures. There is one rule which must be followed if the measurement of performance is to be meaningful. The data collection requirements placed on the operating forces must not be so complicated or extensive that they affect the same performance results which the data is supposed to measure.

The procedures recommended here require only accurate navigation tracks and narratives of the exercise on the part of the participants. The narratives should be of the usual type, containing information such as bearing and distance to all targets detected, weather information, etc.

Following each exercise there is a period of exercise reconstruction during which the participants compare tracks

and determine the results of the exercise. It is during this reconstruction session that information can be gathered from the charts and narratives and transferred to forms which are laid out in an IBM card format. Subsequently, the information is placed on IBM cards and placed in storage to await further processing.

As an example of the method, consider the detection situation where an aircraft radar is the sensor and the target is a submarine snorkel. A recommended format for this combination is shown in Fig. 8.1. The details of this form will be explained further.

One of these forms should be filled out whenever

(a) A contact is made on a true or false target.

(b) The sensor closes to within the predetermined distance, R, of the target and fails to make a detection.

If several unsuccessful passes are made by one sensor, as in Fig. 8.2, the information can be put on one form, otherwise one form should be filled out each time the sensor enters the range circle. If a detection were made on the third pass in Fig. 8.2, two forms would be necessary to record the encounter, one for the two unseccessful passes and one for the detection.

1		0 - Missed opportunity 1 - Detection-correct classif. 2 - Detection-incorrect classif. 3 - False contact
2-5		Time (local)
6-7		Day
8-9		Month
10-11		Year
12		Sea State; 0, 1, 2, 3, 4, 5
13-15		Wind Direction (degrees true)
16-17		Wind Velocity (knots)
18-20		SS Side Number
		Length of opportunity legs in following range band.
21-23		0-5 mile (miles and tenths)
24-26		5-10 mile (miles and tenths)
27-29		10-15 mile (miles and tenths)
30-32		15-20 mile (miles and tenths)
		If detection made:
33-35		True bearing to target
36-38		Range to target (miles and tenths)
39		Type A/C (see key)
40-42		A/C Ground Speed
43		Type Radar (see key)
44		Crew Readiness 0 - 25% 1 - 50% 2 - 75% 3 - 100%

Figure 8.1

Figure 8.2

Several blocks in Fig. 8.1 need further explanation. In block one the term "Detection-incorrect classif." pertains to the situation where an actual target is detected but is classified as a false target. In blocks 21-23 through 30-32 the range circle is subdivided into range bands since the value of R will not necessarily be the same for each radar type. A plexiglass template with range circles marked on it will be useful here. In blocks 39 and 43 it is necessary to refer to a key which assigns an identification number to each type of aircraft and to each type of radar. The crew readiness, noted in block 44, is determined by the evaluator during the exercise reconstruction session.

Upon completetion of an exercise reconstruction, many of these forms will have been accumulated. The results may now be totalled in order to evaluate the sensor. For instance, if R is 20 miles for a particular radar, the value of D in equation (4.1) is equal to the sum of the distances

in blocks 21-23 through 30-32 on those forms which have a zero recorded in block one. The forms which have a two in block one may also be included if it is desired that a missed classification be considered a missed opportunity.

From equation (4.1), the number of missed opportunities is given by

$$M = 2D/\pi R = D/10\pi$$

and from equation (4.2) the probability of detection over a path of width 2R is

$$P = C/(M+C)$$

where C is the number of contacts made, that is, the number of forms which have number one marked in block one. The sweepwidth, W, is given by

$$W = 2PR$$

and is only a figure of merit for the sensor as discussed in Section 5.

The probability of detection under varying operating conditions can be determined by considering only those forms which satisfy the conditions specified. For example, in the aircraft radar vs. snorkel case, the effect of sea state on the probability of detection can be determined by separating the forms according to the sea state determinations in block twelve, and then evaluating P in each case. Similarly, the effects of crew readiness, aircraft speed, time of day, relative bearing to target and relative wind direction can be determined.

After each exercise the information is transferred to

IBM cards, and placed in storage at a computer center. Here, a cumulative record of the performance of sensors under various operating conditions can be kept and the information can be disseminated to interested commands periodically. As the volume of data increases, the resulting performance figures will become more indicative of the true capabilities of the various sensors and the men that operate them.

9. Summary and Conclusions.

A method has been developed, whereby, sensors may be evaluated in an operational environment. It is a simple method, requiring only accurate navigation and routine exercise narratives on the part of the operational forces. The required forms can be filled out by the participants during the post-exercise reconstruction session. By comparing sensor and target tracks and studying the events recorded in the narratives, the following are obtained.

(a) D - the total distance travelled by the sensor within the target's range circle, excluding distance accumulated on successful passes.

(b) C - the number of initial detections made by the sensor.

The number of missed opportunities, $M = 2D/\pi R$ where R is the range beyond which detection is unlikely. It is the radius of the target's range circle. The probability of detection over a path of width 2R is $P = C/(M+C)$ and the resulting lateral range curve is called the modified definite range law. The sweep width $W = 2PR$ is a figure of merit for the sensor's performance. The definite range law, based on sweep width, should not be used to plan search patterns. To do so may result in a parallel track prediction error as high as 13.3% or even higher for small P and small track separation.

The modified definite range law was compared with the more realistic normal and triangular lateral range functions.

The parallel track prediction errors were found to be small and due to the simplicity of the modified law, it is preferred over the other two as a representative lateral range curve.

In Section 8 some procedures were suggested for applying the method to the case of aircraft radar vs. submarine snorkel. The same procedure can be extended to other target-sensor combinations and the results tabulated and disseminated in a form similar to that shown in Fig. 9.1. N = M+C is the number of observations of the sensor under the given conditions. The operation evaluator may wish to modify the format by changing the column headings (operational conditions).

The information can easily be placed on IBM data cards and kept in a computer facility. Summaries, such as in Fig. 9.1, issued periodically by the computer facility will assist commanders in the evaluation of sensors and in the planning of search and screening operations.

When considerable data has been accumulated it becomes a matter of interest to determine the effect of the different operational conditions on the probability of detection. In general, the problem is to determine which conditions significantly affect P. The statistical technique of analysis of variance is applicable here and should be applied when considerable data has been accumulated. It may well be determined that some of the operating conditions listed on forms such as Fig. 8.1 are insignificant and can be dropped

RADAR vs. SUBMARINE

Sensor	Target	Sea State	Aspect	Time share plan	Crew Readiness	R	P	W	N
	(Snorkel) (Surfaced)								

SONAR vs. SUBMARINE

Sensor	Target	Sea State	Sensor Speed	Target Speed	BT Classif.	Crew Readiness	R	P	W	N

Figure 9.1

from further consideration.

Refinement of the modified definite range law is possible. The lateral range function is somehow related to the distribution of initial contact points about the target. This problem was not considered here but it certainly is an area worthy of further study. Computer simulation would undoubtedly provide many answers but an analytical approach may be feasible.

Finally, it must be remembered that a value of P or W is meaningless without an associated statement about the operational conditions and the assumptions under which the particular value was obtained. The development of the modified definite range law assumes that the sensor closes to within a distance, R, of the target. In the case of an alert target, such as a submarine, it may be quite difficult to close to within this range. Therefore, a high value of P in a case such as this is no guarantee that the probability of finding the target in a given area can be made as high as desired by using a clever search plan. The method discussed here is only a start in solving the much larger problem of measuring the effectiveness of search and screening sensors against an alert adversary.

BIBLIOGRAPHY

1. Joseph M. Barron, Random Number Generation on the CDC 1604. United States Naval Postgraduate School, Monterey, California, 1962.

2. E. Parzen, Modern Probability Theory and Its Applications, John Wiley and Sons, Inc., New York, 1960.

APPENDIX A

PATH LENGTH PROGRAM

A program was written to provide the values of $E\left[d\right]$ given in Table 3.1 for various values of c and (d-b). In all runs, the turn angle was assumed to be distributed according to the triangular distribution, (3.1). The length of the search legs was also assumed to be distributed according to the triangular distribution with parameters c and (d-b), (3.2). For each pair of values for c and (d-b), 1000 runs were made. For each run, the following steps were taken:

1. Determine length of first leg of search pattern which will intercept the range circle.
2. Determine x-coordinate of the entry point into the range circle.
3. Determine where along this leg the range circle will be entered.
4. If this leg passes through the range circle, the run is complete and the length of the leg inside the circle is recorded. Start the next run. (Step 1). If leg terminates inside the range circle, record distance inside circle and go to step 5.
5. Generate a turn angle, length of next leg, and determine coordinates of the end of this leg.
6. If this leg terminates inside the circle, record distance and go to Step 5. If the leg terminates

outside the circle, record the distance travelled while inside the circle. Determine the total distance accumulated within the circle on this run, record, and start next run (Step 1).

For each combination of values for c and (d-b), $E\left[d\right]$ is determined by dividing D by 1000 where D is the total distance accumulated within the circle during the 1000 runs.

Fig. A.1 is a flow chart of the program, using the following notation:

RAND - Generate a random number

$\ddot{D}$ - Storage cell used accumulate distance within the circle over 1000 runs

d - Length of present leg within the range circle

N - Cell which records the number of runs made

The random number generator used in this program was tested by Lieutenant Commander Joseph M. Barron, United States Navy, [1], p. 39.

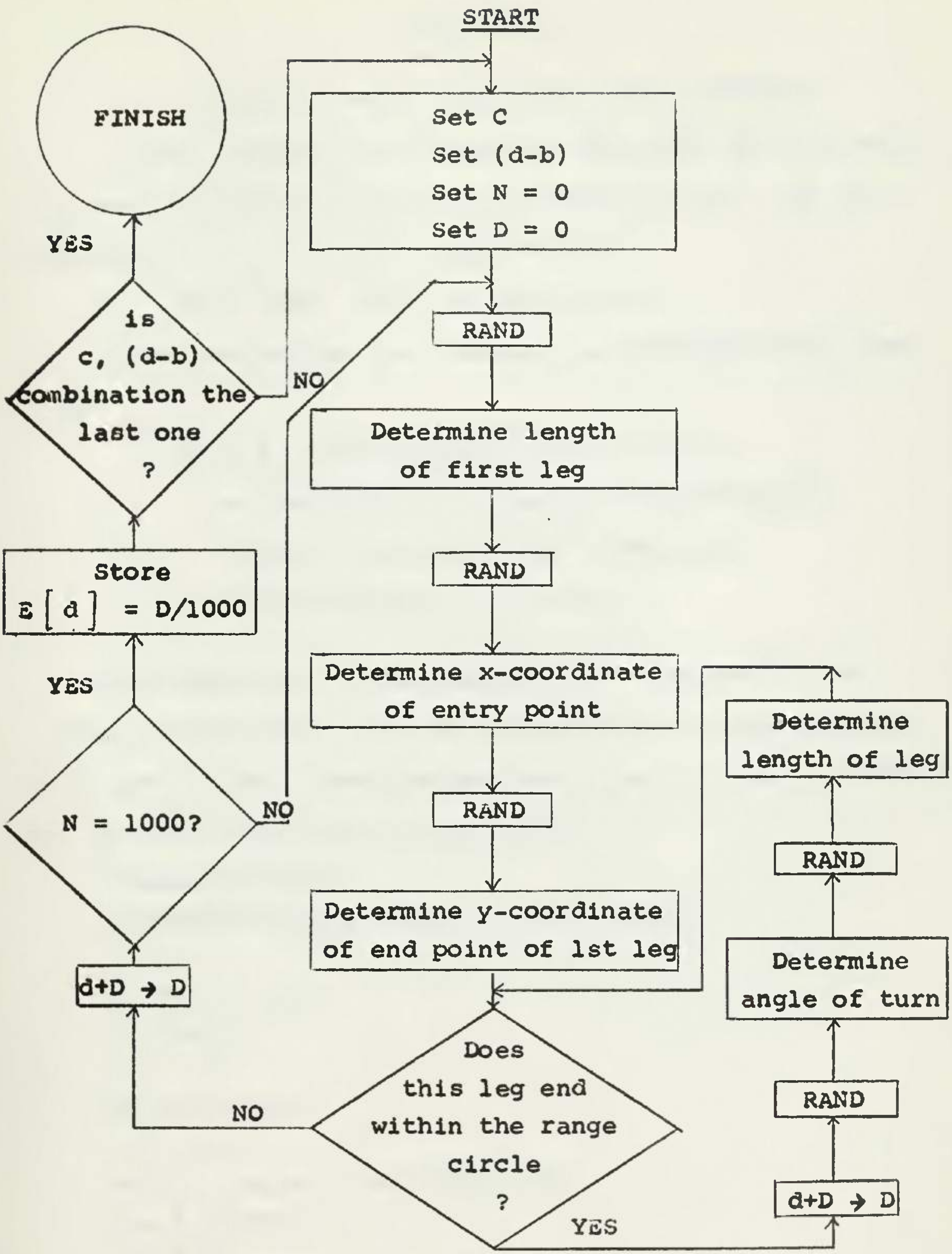

START
FINISH
Set C
Set (d-b)
Set N = 0
Set D = 0
YES
is
c, (d-b)
combination the
last one
?
NO
RAND
Determine length
of first leg
RAND
Store
E [d] = D/1000
YES
Determine x-coordinate
of entry point
Determine
length of leg
N = 1000?
NO
RAND
RAND
Determine y-coordinate
of end point of 1st leg
Determine
angle of turn
d+D → D
Does
this leg end
within the range
circle
?
NO
RAND
YES
d+D → D

Figure A.1

APPENDIX B

PARALLEL SEARCH PREDICTION ERROR PROGRAM

A small program was written to determine Ep, the parallel search prediction error, in equation (7.4). For simplicity, let R = 1. From equation (7.1)

$$\overline{P}m = (4P - 2P^2 + dP^2)/4; \; 0 \leq d \leq 2$$

After evaluating the integrals in equations (7.2) and (7.3)

$$\overline{P}t = P - P^2(8\text{-}12d+6d^2-d^3)/6; \; 1 \leq d \leq 2$$

$$= 2Pd^2+8P^2(d-1)^3/3-(d-1)^2(4P-8P^2+4P^2d)$$

$$-(d-1)\ (8P-4Pd-8P^2+8dP^2)+2Pd(2-d)-$$

$$4P^2(D-D^2+D^3/6) \quad /4; \; 0 \leq d \leq 1$$

The value for P, the probability of detection, was allowed to vary from 0.05 to 0.5 and d, the track spacing, from zero to two. The program itself was written in FORTRAN for the CDC 1604 and is given below.

```
PRØGRAM PRLLERR
DIMENSIØN PT(21), PM(21), E(21), EP(21)
P = O.
DØ 30 L = 1,10
P = P+.05
D = -.1
DØ 20 J=0,10
D = D+.1
PM(J) = (4.*P-2.*P*P+D*P*P)/4.
A = 2.*P*D*D
B = D-1.
C = B*B*B*P*P*8./3.+B*B*(8.*P*P-4.*P-4*P*P*D)
```

```
      CC = B*(4.*P*D-8.*P+8.*P*P-8.*D*P*P)
      H = 2.*P*D*(2.-D)-4.*P*P*(D-D*D-D*D*D/6.)
      PT(J) = (A+C+CC+H)/4.
      E(J) = PT(J)-PM(J)
      EE(J) = PT(J)+PM(J)
      EP(J) = 2.*E(J)/EE(J)
      PRINT 20, PT(J),PM(J),E(J),EP(J)
   20 FØRMAT (4F10.7)
      D = 1.
      DØ 30 J = 11,20
      D = D+.1
      PT(J) = P-P*P*(8.-12.*D+6.*D*D-D*D*D)/6.
      PM(J) = (4.*P-2.*P*P+D*P*P)/4.
      E(J) = PT(J)-PM(J)
      EE(J) = PT(J)+PM(J)
      EP(J) = 2.*E(J)/EE(J)
      PRINT 30, PT(J), PM(J), E(J), EP(J)
   30 FØRMAT (4F10.7)
      END
      END
```

The program results are arranged below in Table B.1. Note that a maximum occurs at d = 1.3R for all values of P. At this track spacing $\bar{P}t-\bar{P}m$ is a maximum, however, $\bar{P}t$ and $\bar{P}m$ each has its maximum at d = 2R. Here, both $\bar{P}t$ and $\bar{P}m$ are equal to P. Another maximum for Ep occurs at d = 0.1. Here, $\bar{P}m-\bar{P}t$ is a maximum.

d \ P	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
0	.009	.018	.027	.038	.049	.061	.073	.087	.102	.118
.1	.009	.019	.030	.041	.053	.066	.080	.094	.110	.127
.2	.009	.019	.029	.041	.052	.065	.078	.092	.107	.124
.3	.008	.017	.027	.037	.047	.058	.070	.083	.097	.111
.4	.007	.015	.022	.031	.039	.048	.058	.068	.080	.091
.5	.005	.011	.017	.023	.029	.036	.043	.050	.058	.066
.6	.003	.007	.010	.014	.018	.022	.026	.031	.035	.040
.7	.001	.002	.004	.005	.006	.008	.009	.011	.012	.014
.8	.001	.002	.003	.004	.005	.006	.007	.008	.009	.010
.9	.003	.006	.008	.011	.014	.017	.021	.024	.027	.031
1.0	.004	.009	.013	.017	.022	.027	.031	.036	.041	.047
1.1	.005	.011	.016	.021	.027	.033	.039	.044	.051	.057
1.2	.006	.012	.018	.024	.030	.036	.042	.049	.055	.062
1.3	.006	.012	.018	.024	.030	.036	.043	.049	.056	.063
1.4	.006	.012	.017	.023	.029	.035	.041	.047	.054	.060
1.5	.005	.010	.016	.021	.027	.032	.037	.043	.048	.054
1.6	.004	.009	.014	.018	.023	.027	.032	.037	.041	.046
1.7	.004	.007	.011	.014	.018	.021	.025	.029	.032	.036
1.8	.002	.005	.007	.010	.012	.015	.017	.020	.022	.025
1.9	.001	.002	.004	.005	.006	.007	.009	.010	.011	.012
2.0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

Values of Ep/100 for various values of P and d

Table B.1